

C-planarity of Embedded Cyclic c-Graphs

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Abstract. We show that c-planarity is solvable in quadratic time for flat clustered graphs with three clusters if the combinatorial embedding of the underlying graph is fixed. In simpler graph-theoretical terms our result can be viewed as follows. Given a graph G with the vertex set partitioned into three parts embedded on a 2-sphere, our algorithm decides if we can augment G by adding edges without creating an edge-crossing so that in the resulting spherical graph the vertices of each part induce a connected sub-graph. We proceed by a reduction to the problem of testing the existence of a perfect matching in planar bipartite graphs. We formulate our result in a slightly more general setting of cyclic clustered graphs, i.e., the simple graph obtained by contracting each cluster, where we disregard loops and multi-edges, is a cycle.

1 Introduction

Testing planarity of graphs with additional constraints is a popular theme in the area of graph visualizations. One of most the prominent such planarity variants, c-planarity, raised in 1995 by Feng, Cohen and Eades [12,13] asks for a given planar graph G equipped with a hierarchical structure on its vertex set, i.e., clusters, to decide if a planar embedding G with the following property exists: the vertices in each cluster are drawn inside a disc so that the discs form a laminar set family corresponding to the given hierarchical structure and the embedding has the least possible number of edge-crossings with the boundaries of the discs. Shortly after, several groups of researchers tried to settle the main open problem formulated by Feng et al. asking to decide its complexity status, i.e., either provide a polynomial/sub-exponential-time algorithm for c-planarity or show its NP-hardness. First, Biedl [5] gave a polynomial-time algorithm for c-planarity with two clusters. A different approach for two clusters was considered by Hong and Nagamochi [19] and quite recently in [15]. The result also follows from a work by Gutwenger et al. [17]. Beyond two clusters a polynomial time algorithm for c-planarity was obtained only in special cases, e.g., [8,16,17,20,21], and most recently in [6,7]. Cortese et al. [9] shows that c-planarity is solvable in

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** The omitted parts of the proof are in the appendix.

polynomial time if the underlying graph is a cycle and the number of clusters is at most three.

In the present work we generalize the result of Cortese et al. to the class of all planar graphs with a given combinatorial embedding. In a recent pre-print [14] we established a strengthening for trees, where we do not fix the embedding. In the general case (including already the case of three clusters) of so-called flat clustered graphs a similar result was obtained only in very limited cases. Specifically, either when every face of G is incident to at most five vertices [10,15], or when there exist at most two vertices of a cluster incident to a single face [7]. We remark that the techniques of the previously mentioned papers do not give a polynomial-time algorithm for the case of three clusters, and also do not seem to be adaptable to this setting. Our result and the technique used to achieve it suggest that, for a fairly general class of clustered graphs, c-planarity could be tractable/solvable in sub-exponential time at least with a fixed combinatorial embedding.

Notation. Let $G = (V, E)$ denote a connected planar graph possibly with multi-edges. For standard graph theoretical definitions such as path, cycle, walk etc., we refer reader to [11, Section 1]. A *drawing* of G is a representation of G in the plane where every vertex in V is represented by a unique point and every edge $e = uv$ in E is represented by a Jordan arc joining the two points that represent u and v . We assume that in a drawing no edge passes through a vertex, no two edges touch and every pair of edges cross in finitely many points. An *embedding* of G is an edge-crossing free drawing. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words “vertex” and “edge” in both contexts. A *face* in an embedding is a connected component of the complement of the embedding of G (as a topological space) in the plane. The *facial walk* of f is the closed walk in G with a fixed orientation that we obtain by traversing the boundary of f counter-clockwise. In order to simplify the notation we sometimes denote the facial walk of a face f by f . A pair of consecutive edges e and e' in a facial walk f creates a *wedge* incident to f at their common vertex. A vertex or an edge is *incident* to a face f , if it appears on its facial walk. The *rotation* at a vertex is the counter-clockwise cyclic order of the end pieces of its incident edges in a drawing of G . An embedding of G is up to an isotopy and the choice of an *outer* (unbounded) face described by the rotations at its vertices. We call such a description of an embedding of G a *combinatorial embedding*. Remaining faces are *inner faces*. The *interior* and *exterior* of a cycle in an embedded graph is the bounded and unbounded, respectively, connected component of its complement in the plane. Similarly, the *interior* and *exterior* of an inner face in an embedded graph is the bounded and unbounded, respectively, connected component of the complement of its facial walk in the plane, and vice-versa for the outer face. When talking about interior/exterior or area of a cycle in a graph G with a combinatorial embedding and a *designated* outer face we mean it with respect to an embedding in the isotopy class that G defines. For $V' \subseteq V$ we denote by $G[V']$ the sub-graph of G induced by V' .

A *flat clustered graph*, shortly *c-graph*, is a pair (G, T) , where $G = (V, E)$ is a graph and $T = \{V_0, \dots, V_{c-1}\}$, $\bigsqcup_i V_i = V$, is a partition of the vertex set into *clusters*. See Figure 1 for an illustration. A c-graph (G, T) is *clustered planar* (or briefly *c-planar*) if G has an embedding in the plane such that (i) for every $V_i \in T$ there is a topological disc $D(V_i)$, where $\text{interior}(D(V_i)) \cap \text{interior}(D(V_j)) = \emptyset$, if $i \neq j$, containing all the vertices of V_i in its interior, and (ii) every edge of G intersects the boundary of $D(V_i)$ at most once for every $D(V_i)$. A c-graph (G, T) with a given combinatorial embedding of G is *c-planar* if additionally the embedding is combinatorially described as given. A *clustered drawing and embedding* of a flat clustered graph (G, T) is a drawing and embedding, respectively, of G satisfying (i) and (ii). In 1995 Feng, Cohen and Eades [12,13] introduced the notion of clustered planarity for clustered graphs, shortly c-planarity, (using, a more general, hierarchical clustering) as a natural generalization of graph planarity. (Under a different name Lengauer [22] studied a similar concept in 1989.)

By slightly abusing the notation for the rest of the paper G denotes a flat c-graph $(G, T) = (V_0 \sqcup V_1 \sqcup \dots \sqcup V_{c-1}, E)$ with c clusters V_0, V_1, \dots and V_{c-1} , and a given combinatorial embedding, and we assume that G is *cyclic* [15, Section 6]. Thus, every $e = uv$ of G is such that $u \in V_i$ and $v \in V_j$ where $j - i \bmod c \leq 1$ and for every i there exists an edge in G between V_i and $V_{i+1 \bmod c}$. In the case of three clusters, the first condition is redundant. If the second condition is violated, the problem was essentially

solved for three clusters as discussed in Section 2.3. We assume that G is connected, since in the problem that we are studying, the connected components of G can be treated separately. Indeed, without loss of generality we assume throughout the paper that in a clustered embedding of G the clusters are unbounded wedges defined by pairs of rays emanating from the origin (see Figure 2a) that is disjoint from all the edges (see Appendix). We call such a clustered drawing a *fan drawing*.

Thus, a connected component in a clustered embedding can be drawn so that it is disjoint from a ball B centered at the origin of radius $\epsilon > 0$ for any ϵ . The rest of the graph is then embedded inductively inside B . The aim of the present work is to prove the following.

Theorem 1. *There exists a quadratic-time algorithm in $|V(G)|$ to test if a cyclic c-graph (G, T) is c-planar.*

Further research directions. We think that our technique should be extendable by means of Euler's formula to resolve the c-planarity in more general

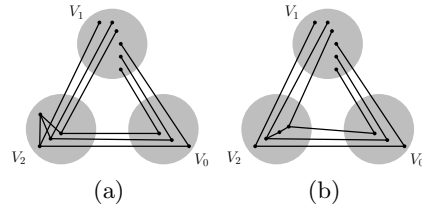


Fig. 1: A c-graph that is not c-planar (left); and a c-planar c-graph (right).

situations than the one treated in the present paper. In particular, we suspect that the technique should yield a generalization of the characterization of strip planar clustered graphs [14, Section 5]. That would allow us to work with graphs without a fixed embedding. We mention that the tractability in a special case of our problem known as cyclic level planarity, when the embedding is not fixed, follows from a recent work of Angelini et al. [2].

Organization. In Section 2 we introduce concepts used in the proof of our result. We give an outline of our approach in Section 2.1. A more detailed description and a proof of correctness of our algorithm is in Section 3.

2 Preliminaries

2.1 Outline of the approach

By [13, Theorem 1] deciding c-planarity of instances G in which all $G[V_i]$'s are connected amounts to checking if an outer face of G can be chosen so that every V_i is embedded in the outer face of $G[V \setminus V_i]$. On the other hand, once we have a clustered embedding of G we can augment G by adding edges drawn inside clusters without creating an edge-crossing so that clusters become connected. These observations suggest that c-planarity of G could be viewed as a connectivity augmentation problem, for example as in [7,15], in which we want to decide if it is possible to make clusters connected while maintaining the planarity of G . One minor problem with this viewpoint is the fact that if G is c-planar we do not allow a cluster V_i to induce a cycle such that clusters V_j and $V_{j'}$, $i \neq j, j'$, are drawn on its opposite sides. However, this cannot happen if G is cyclic. Following the above line of thought our algorithm tries to augment G by subdividing its faces with paths and edges. We proceed in two steps. In the first step, Section 3.2, we either detect that G is not c-planar or similarly as in [1] and [14] by turning clusters into independent sets and adding certain paths we normalize the instance. In the second step, Section 3.1, we decide if the normalized instance can be further augmented by edges as desired.

In order to prove the correctness of the second step of the algorithm we use the notion of the *winding number* $\text{wn}(W) \in \mathbb{Z}$ of a walk W of G , as defined in Section 2.3. The parameter $\text{wn}(W)$ says how many times and in which sense a walk W of G winds around the origin in a clustered drawing of G . Thus, G is not c-planar if there exists a face f such that for its facial walk $|\text{wn}(f)| > 1$ or if there exists at least two inner faces f with $|\text{wn}(f)| > 0$. However, it can be easily seen that this necessary condition of c-planarity is not sufficient except when G is a cycle [9]. The necessary condition allows us to reduce the c-planarity testing problem of a normalized instance to the problem of finding a perfect matching in an auxiliary face-vertex incidence graph which is polynomially solvable. The novelty of our work lies in the use of the winding number in the context of connectivity augmentation guided by the flow and matching in the auxiliary face-vertex incidence graph à la [1] and [14], respectively.

We remark that the approach of [1] via a variant of upward embeddings for directed graphs in our settings has several problems that seem quite hard to

overcome, the main one being the fact that the result of Bertolazzi et al. [4] does not extend, at least not in a natural way, to the drawings on the rolling cylinder, see e.g., Auer et al.[3] for the definition of these drawings. We are not aware of a polynomial-time algorithm for the corresponding problem, nor a corresponding NP-hardness result, and find the corresponding algorithmic question interesting and related to our problem.

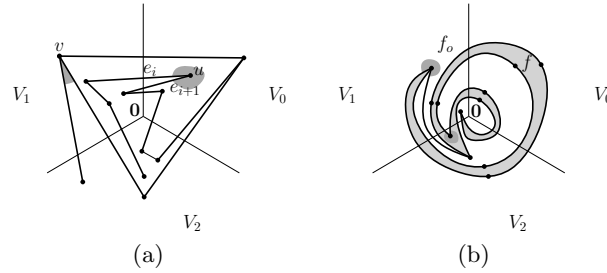


Fig. 2: (a) A clustered graph $G = (V_0 \uplus V_1 \uplus V_2, E)$ with clusters represented by wedges bounded by rays meeting at the origin. The highlighted wedge at u is concave and at v convex. (b) A semi-simple face f and the outer face f_o with an incident concave wedge.

2.2 Winding number

We define the winding number $\text{wn}(W)$ of a closed oriented walk W in a drawing disjoint from the origin of a graph G (possibly with crossings). In what follows facial walks are understood with the orientations as in an embedding of G with the given rotations and a face f_o being a designated outer face. By viewing a closed walk W in the drawing as a continuous function w from the unit circle S^1 to $\mathbb{R}^2 \setminus \mathbf{0}$, the winding number $\text{wn}(W) \in \mathbb{Z}$ corresponds to the element of the fundamental group of S^1 [18, Chapter 1.1] represented by $\frac{w(x)}{\|w(x)\|_2}$. Let W_1 and W_2 denote a pair of oriented closed walks meeting in a vertex v . Let W denote the closed oriented walk from v to v obtained by concatenating W_1 and W_2 . By the definition of wn we have $\text{wn}(W) = \text{wn}(W_1) + \text{wn}(W_2)$. Let f_1 and f_2 , $f_o \neq f_1, f_2$, denote a pair of faces of G whose walks intersect in a single walk. Let G' denote a graph we get from G by deleting edges incident to both f_1 and f_2 . Let f denote the new face thereby obtained. Since f_1 and f_2 intersect in a single walk, the boundary of f is connected. In the drawing of G' inherited from the drawing of G we have $\text{wn}(f) = \text{wn}(f_1) + \text{wn}(f_2)$, since common edges of f_1 and f_2 are traversed in opposite directions by f_1 and f_2 . A face or a vertex is in the interior of a closed walk W in G if it is in the interior of a cycle induced by the edges of W in an embedding of G with the given rotations and f_o as the outer face. The previous observation is easily generalized by a simple inductive

argument as follows

$$(*) \quad \sum_f \text{wn}(f) = \text{wn}(W)$$

where we sum over all faces f of G in the interior of the closed walk W in G . In particular, $\sum_f \text{wn}(f) = \text{wn}(f_o)$, where we sum over all faces $f \neq f_o$ of G .

2.3 Labeling vertices

Let $\gamma : V \rightarrow \{0, 1, \dots, c-1\}$ be a labeling of the vertices V by integers such that $\gamma(v) = i$ if $v \in V_i$. Let W denote an oriented closed walk in a clustered drawing of G . We put $\text{height}(W) = \sum_{v'u' \in E(W)} g(\gamma(u') - \gamma(v'))$, where $g(0) = 0$, $g(1) = g(1-c) = 1$ and $g(c-1) = g(-1) = -1$. We have the following.

Lemma 1. *For a walk W in a fan drawing of G we have $\text{wn}(W) = \text{height}(W)/c$.*

Proof. The number of times the walk W crosses the ray between V_i and $V_{i+1 \bmod c}$ from right to left w.r.t. to the direction of the ray is $\text{wn}_i^+(W) = \sum_{v'u' \in E(W)} g(\gamma(u') - \gamma(v'))$, where we sum over the edges $v'u'$ in the walk W , where $v' \in V_i$ immediately precedes $u' \in V_{i+1 \bmod c}$ in the walk. Similarly, we define $\text{wn}_i^-(W) = \sum_{v'u' \in E(W)} g(\gamma(u') - \gamma(v'))$, where we sum over the edges $v'u'$ in W , where $v' \in V_{i+1 \bmod c}$ immediately precedes $u' \in V_i$ in the walk. We have, $\text{wn}(W) = \text{wn}_i^+(W) + \text{wn}_i^-(W)$ which in turn implies $c \cdot \text{wn}(W) = \sum_i (\text{wn}_i^+(W) + \text{wn}_i^-(W)) = \text{height}(W)$. ■

By the previous lemma $\text{wn}(W)$ is determined already by the c -graph G and is the same in all clustered drawings of G , and hence, putting $\text{wn}(W) := \text{height}(W)/c$, for a walk W with a fixed orientation, allows us to speak about $\text{wn}(W)$ without referring to a particular drawing of G . Thus, $\text{wn}(W)$ tells us the winding number of W in any clustered drawing. By Jordan-Schönflies theorem G the following holds.

Lemma 2. *G is not c -planar if there exists a face f such that $|\text{wn}(f)| > 1$ or if there exists more than one inner face f' with $|\text{wn}(f')| = 1$.*

Proof. In a crossing free drawing $|\text{wn}(f)| \leq 1$ for every face f . If $|\text{wn}(f')| = 1$ the origin $\mathbf{0}$ lies in the interior of f' since otherwise the facial walk is null-homotopic, i.e., homotopic to a constant map, in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ (contradiction). However, interiors of faces are disjoint. ■

If $\text{wn}(f) = 0$ for all faces f , [14, Lemma 1.2] extends easily to this case, reducing the problem to the work of Angelini et al. [1]. Thus, by Lemma 2 and for the sake of simplicity of the presentation, throughout the paper we assume that there exists a pair of faces f_o, f'_o , $\text{wn}(f_o) = \text{wn}(f'_o) \neq 0$ (by $(*)$ there cannot be just one such face) one of which, let's say f_o , we designate as an *outer face*. The roles of f_o and f'_o are, in fact, interchangeable. Also such a restriction is by no means crucial in our problem, and alternatively, it is always possible to choose and subdivide the outer face in the normalized instance (defined later) by a path so that the restriction is satisfied.

Viewing a facial walk f as a sequence of vertices and edges $w_0 e_0 w_1 e_2 \dots e_m w_m$, where $e_{i-1} = w_{i-1} w_i$, let V_f be the set $\{w_0, \dots, w_m\}$ of *vertex occurrences* along f . We treat V_f also as a multi-set of vertices, and thus, γ is defined on its elements. Let $\gamma_f : V_f \rightarrow \mathbb{N}$, for $f \neq f_o, f'_o$, be a labeling of the elements of V_f by integers defined as follows. We mark all the vertex occurrences in V_f as unprocessed. We pick an arbitrary vertex occurrence $v \in V_f$, set $\gamma_f(v) := \gamma(v)$ and mark v as processed. We repeatedly pick an unprocessed vertex occurrence $u \in V_f$ that has its predecessor or successor v along the boundary walk of f in V_f processed. We put $\gamma_f(u) := \gamma_f(v) + g(\gamma(u) - \gamma(v))$. Intuitively, γ_f records the distance in terms of “winding around origin” of vertex occurrences along the boundary walk of f from a single chosen vertex occurrence. Since $\text{wn}(f) = 0$ the function $\gamma_f(u)$ is completely determined by the choice of the first occurrence of a vertex we processed. This choice is irrelevant for our use of γ_f as we see later. Also notice that $\gamma(v) = \gamma_f(v) \pmod c$ for all vertices incident to f .

A normalized instance allows only the faces of the types defined next. An element v in V_f is a *local minimum* (*maximum*) of a face f if in the facial walk f the value of $\gamma(v)$ is not bigger (not smaller) with respect to the relation $0 < 1 < \dots < c-1 < 0$ than the value of its successor and predecessor. A walk W in G is (*strictly*) *monotone with respect to γ* if the labels of the occurrences of vertices on W form a (strictly) monotone sequence with respect to the relation $0 < 1 < \dots < c-1 < 0$ when ordered in the correspondence with their appearance on W . The face f is *simple* if f has at most one local minimum. It follows that a simple face f has also at most one local maximum. The inner face $f \neq f'_o$ is *semi-simple* if f has exactly two local minima and maxima and these minima and maxima, respectively, have the same γ_f value.

3 Algorithm

A cyclic c-graph G is *normalized* if

- (i) G is connected;
- (ii) each cluster V_i induces an independent set; and
- (iii) each face of G is simple or semi-simple, and f_o and f'_o are both simple.

Suppose that (i)–(iii) are satisfied. By (ii) we put directions on all the edges in G as follows. Let \vec{G} denote the directed c-graph obtained from G by orienting every edge uv from the vertex with the smaller label $\min(\gamma(u), \gamma(v))$ to the vertex with the bigger label $\max(\gamma(u), \gamma(v))$ with respect to the relation $0 < 1 < \dots < c-1 < 0$. A *sink* and *source* of \vec{G} is a vertex with no outgoing and incoming, respectively, edges.

Let e denote an edge of G not contained in a single cluster. Given a clustered embedding \mathcal{D} of G let $\mathbf{p}_e := \mathbf{p}_e(\mathcal{D})$ denote the intersection point of e with a ray separating a pair of clusters. Let e_0, \dots, e_{k-1} be the edges incident to a sink or source u . By Jordan curve theorem it is not hard to see that (i)–(iii) imply that a clustered embedding \mathcal{D} of G is “combinatorially” determined once we order the set of intersection points $\mathbf{p}_{e_0}, \dots, \mathbf{p}_{e_{k-1}}$ along rays separating clusters for every sink and sources u in G . Moreover, the set of intersection points corresponding

to a sink or source u admits in an embedding only orders that are cyclic shifts of one another, since we have the rotations at vertices of G fixed. The wedge in \mathcal{D} formed by a pair of edges e_i and e_{i+1} incident to a face f at its local extreme u is *concave* (see Figure 2a for an illustration) if u is a sink or source of \vec{G} and the line segment $\mathbf{p}_{e_i}\mathbf{p}_{e_{i+1}}$ contains all the other points \mathbf{p}_{e_j} or in other words the order of intersection points corresponding to u is $\mathbf{p}_{e_{i+1}}, \mathbf{p}_{e_{i+2}}, \dots, \mathbf{p}_{e_{k-1}}, \mathbf{p}_{e_0}, \dots, \mathbf{p}_{e_i}$. A non-concave wedge is *convex*. Note that in \mathcal{D} every sink or source is incident to exactly one concave wedge that in turn determines the order of intersection points. Thus, combinatorially \mathcal{D} is also determined by a prescription of concave wedges at sink and sources.

Let S be the set of sinks and sources of \vec{G} . Let F denote the union of the set of semi-simple faces of G with a subset of $\{f_o, f'_o\}$ containing faces incident to a sink and a source. We construct a planar bipartite graph $I = (S \cup F, E(I))$ with parts S and F , where $s \in S$ and $f \in F$ is joined by an edge if s is incident to f . Given that (i)–(iii) are satisfied, the existence of a perfect matching M in I is a necessary condition for G being c-planar. Indeed, as we just said, in a clustered embedding, each source or sink has exactly one of its wedges concave. On the other hand, by Jordan curve theorem it can be easily checked that in the clustered embedding

- (A) every semi-simple face is incident to exactly one concave wedge
- (B) faces f_o and f'_o are incident to one concave wedge if they are incident to a sink and source, and
- (C) all the other faces are not incident to any concave wedges at the minimum and maximum.

This is fairly easy to see if G is vertex two-connected, see Figure 2b for an illustration. The cycle C corresponding to a closed walk is obtained by traversing the walk and introducing a new vertex for each vertex occurrence in the walk. For a face f incident to cut-vertices, (A)–(C) follows by considering the cycle corresponding to the facial walk of f (treated as a face) embedded in a close vicinity of the boundary of f . Thus, a desired matching M is obtained by matching each source or sink with the face incident to its concave wedge.

We show in Section 3.1 that if M exists G is c-planar by augmenting G with edges as described in Section 2.1. Testing the existence, but even counting perfect matchings in a planar bipartite graph can be carried out in a polynomial time [23, Section 8].

The running time of our algorithm is $O(|V|^2)$ since finding the perfect matching can be done in $O(|V|^2)$ time, due to $|E(I)| = O(|V|)$, and the pre-processing step including the construction of I and the normalization will be easily seen to have this time complexity. Also computing the winding number for all the faces can be performed in a linear time by Lemma 1. First, we explain and prove the correctness of the algorithm for instances satisfying (i)–(iii). In Section 3.2, we show a polynomial-time reduction of the general case to instances satisfying (i)–(iii). We often use Jordan–Schönflies theorem without explicitly mentioning it.

3.1 Constructing a clustered embedding

Given a normalized instance G and a matching M between sources and sinks in S , and faces in F of G we construct a clustered embedding of G as follows. Recall that we assume that G does not have a face f with $|\text{wn}(f)| > 0$ besides f_o and f'_o . We start with \vec{G} defined above and add edges to it thereby eliminating all the sinks and sources,

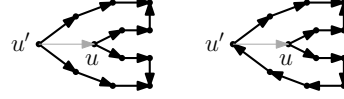


Fig. 3: Subdividing a semi-simple face (left). Subdividing a simple face f'_o (right).

see Figure 3. Let $u \in S$ be a source matched in M with f . If f is a semi-simple inner face let u' denote another local minimum incident to f . We add to \vec{G} an edge $\vec{u'u}$ embedded in the interior of f . If $f = f_o$ or $f = f'_o$ we join u by $\vec{u'u}$ with the vertex in the same cluster u' so that we subdivide f into two simple faces f' and f'' such that $\text{wn}(f') = 0$ and $\text{wn}(f'') = \text{wn}(f)$. If $f = f_o$ face f'' is the new outer face. By Lemma 1, such a vertex u' exists and it is unique.

We proceed with $u \in S$ that are sinks analogously thereby eliminating all the sinks and source in the resulting graph \vec{G}' , where by G' we denote its underlying undirected graph. By Lemma 1, there still exists exactly one inner face f'_o with a non-zero winding number in the resulting graph G' .

Lemma 3. G' has exactly one inner face f'_o such that $|\text{wn}(f'_o)| = 1$.

Since $\gamma(v) = \gamma_f(v) \pmod{c}$ for every face $f \neq f_o, f'_o$ and v incident to f , every edge we added joins a pair of vertices in the same cluster.

Lemma 4. The induced sub-graph $G'[V_i]$ of (undirected) G' does not contain a cycle for $i = 0, 1, \dots, c-1$.

Proof. For the sake of contradiction suppose that a cycle C is contained in $G'[V_{j'}]$. Let us choose C such that the area of its interior is minimized. Since $G[V_{j'}]$ is an independent set all the edges of C are newly added. Thus, by looking at the rotation of an arbitrary vertex v' of C we see that v' is incident to a vertex v from V_j , $j \neq j'$, in the interior of C . Indeed, no two edges of C subdivide the same face of G .

Using the fact that \vec{G}' does not contain any source or sink, we show that a vertex w in the interior of C belongs to an oriented cycle C' (by chance also directed in \vec{G}'), whose interior is contained in the interior of C such that $\text{wn}(C') > 0$. The cycle C' is obtained by following a directed path in \vec{G}' (from which it inherits its orientation) passing through v . Either both ends of the path meet each other, they both meet C , or the path meet itself in the interior. In the first two cases we can take $w := v$ in the last case it can happen that the directed path gives rise to a cycle C' not containing v . However, C' is not induced by a single cluster by the choice of C , and thus, $\text{wn}(C') > 0$ by Lemma 1 and C'

contains a vertex w from V_j . Let F' denote the set of faces in the interior of C and not in the interior of C' . In all cases it can be seen by Lemma 1 that $\text{wn}(C') > 0$.

Indeed, as we proved in the proof of Lemma 1 $\text{wn}(C') = \text{wn}_j^+(C') + \text{wn}_j^-(C')$. Since C' follows a directed path and is not induced by a single cluster we have $\text{wn}_j^+(W) > 0$ and $\text{wn}_j^-(W) = 0$. Hence, $\text{wn}(C') = \text{wn}_j^+(C') + \text{wn}_j^-(C') > 0$.

By (*) it follows that C' contains the unique inner face with a non-zero winding number in its interior. Then Lemma 3 with (*) yields the following contradiction

$$0 = \text{wn}(C) = \text{wn}(C') + \sum_{f \in F'} \text{wn}(f) = \text{wn}(C') \neq 0$$

■

Let $E' \subseteq \bigcup_i \binom{V_i}{2} \setminus E(G')$ such that each edge in E' can be added to the embedding of G' without creating a crossing or increasing the number of inner faces with a non-zero winding number. We do not put any direction on the edges in E' . Since every inner face $\neq f_o$ in G' is simple, and its outer face and the face f_o are not adjacent to a source or sink, all the edges in E' can be introduced simultaneously without creating a crossing. In particular, no edge of E' subdivides f_o or the outer face. Let E'' denote a maximal subset of E' that does not introduce a cycle in $(G' \cup E'')[V_i]$ for every $i = 0, 1, \dots, c-1$ (see Figure 4), where $G' \cup E'' = (V(G'), E(G') \cup E'')$. By Lemma 4, E'' is well-defined.

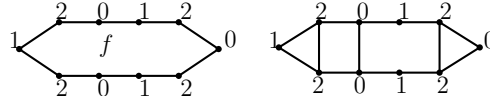


Fig. 4: A simple face f of G' (left). The face f subdivided with edges of E'' (right). Labels at vertices are their γ values (or indices of their clusters).

Lemma 5. $(G' \cup E'')[V_i]$ is a tree for $i = 0, 1, \dots, c-1$.

Proof. Suppose for the sake of contradiction that $(G' \cup E'')[V_i]$ for some i is not a tree, and thus, it is just a forest with more than one connected component. It follows that either (1) there exists a cycle in $(G' \cup E'')[V \setminus V_i]$ containing a vertex v of V_i in its interior or (2) a pair of vertices of V_i in different connected components of $(G' \cup E'')[V_i]$ are incident to the same face of $(G' \cup E'')$. The claim (1) or (2) implies that there exists a cycle C in $(G' \cup E'')[V \setminus V_i]$ containing a vertex w of V_i in its interior. Similarly as in the proof of Lemma 4, by following a directed path through v we obtain an oriented cycle C' (this time not necessarily directed) in G , whose interior is contained in the interior of C with $\text{wn}(C') > 0$ yielding a contradiction.

Indeed, as we proved in the proof of Lemma 1 $\text{wn}(C') = \text{wn}_i^+(C') + \text{wn}_i^-(C')$. Since C' is not induced by a single cluster and follows in the interior of C a directed path, and C does not have any vertex in V_i we have $\text{wn}_i^+(W) > 0$ and $\text{wn}_i^-(W) = 0$. Hence, $\text{wn}(C') = \text{wn}_i^+(C') + \text{wn}_i^-(C') > 0$. ■

By Lemma 5, every F_i is a tree. Taking a close neighborhood of each such F_i as a disc representing the cluster V_i we obtain a desired clustered embedding of $(G' \cup E'')$. In the obtained embedding we just delete edges not belonging to G and that concludes the proof of the correctness of our algorithm.

3.2 Normalization

In the present section we normalize the instance so that (i)-(iii) are satisfied. We argued the connectedness in Introduction, and hence, (i) is taken care of. To achieve (ii) is fairly standard by contracting components induced by clusters to vertices. Thus, it remains to satisfy (iii).

We want to sub-divide a non-simple face f into a pair of faces one of which is semi-simple by a monotone path P' w.r.t. γ . Let uPv denote an oriented monotone sub-walk of f w.r.t. γ joining a local minimum u and maximum v of f minimizing $|\text{height}(P)|$. Let vQv' denote the oriented monotone walk with $|\text{height}(P)| = |\text{height}(Q)|$ immediately following P on the facial walk of f , and let $u'Q'u$ be such walk immediately preceding P on the facial walk of f . Note that Q and Q' exists due to the minimality of P and that we have $\text{height}(Q) = \text{height}(Q') = -\text{height}(P)$. Similarly as in [14] we subdivide f into two faces f' and f'' by a strictly monotone path $v'P'u'$ w.r.t. γ . Hence, $\text{height}(P) = \text{height}(P')$. We have $\text{height}(Q) = \text{height}(Q') = -\text{height}(P) = -\text{height}(P')$. Thus, by Lemma 1 if f with $\text{wn}(f) \neq 0$ is semi-simple we obtain a simple face f' with $\text{wn}(f') \neq 0$ and a semi-simple face f'' with $\text{wn}(f'') = 0$ as desired. Indeed, $\text{wn}(f'') = \text{height}(P') + \text{height}(Q') + \text{height}(P) + \text{height}(Q) = 0$ and $c \cdot \text{wn}(f) = \text{height}(v'P'u') + \text{height}(Q') + \text{height}(P) + \text{height}(Q) = \text{height}(v'P'u') - \text{height}(P') = c \cdot \text{wn}(f')$. It remains to show the following lemma, since both f' and f'' are incident to less local minima and maxima than f if f is not semi-simple. Hence, after $O(|V|)$ facial subdivisions we obtain a desired instance, since $|E(I)| = O(|V|)$.

Lemma 6. *If the c -graph G is c -planar then by subdividing f of G by P' into a pair of faces f' and f'' , where f'' is semi-simple we obtain a c -planar c -graph. Moreover, $\text{wn}(f') = \text{wn}(f)$ and $\text{wn}(f'') = 0$.*

Proof. The second statement is proved above. Hence, we deal just with the first one. Let e_u and e'_u denote the first edge on P and the last edge on Q' , respectively. Let e_v and e'_v denote the last edge on P and the first edge on Q , respectively. Let $e_{v'}$ and $e_{u'}$ denote the last edge on Q and the first edge on Q' . Let $\mathbf{p}_u = \mathbf{p}_{e_u}$, and $\mathbf{p}_{v'} = \mathbf{p}_{e_{v'}}$ denote the intersection of the edges e_u , and $e_{v'}$, respectively, with a ray separating a pair of clusters. Let ω_u and ω_v denote the wedge between e_u, e'_u and e_v, e'_v , respectively, in f .

We presently show that subdividing f with P' preserves c-planarity, since a clustered embedding without P' can be deformed so that P' can be added to a clustered planar embedding without creating a crossing, while keeping the embedding clustered. This is not hard to see if, let's say ω_v , is convex and the line segment $\mathbf{p_u p_{v'}}$ is not crossed by an edge. Since ω_v is convex, the relative interior of $\mathbf{p_u p_{v'}}$ is contained in the interior of f . Note that $u'Q'PQv'$ is a sub-walk of f since f is not simple. We draw a curve C joining u' with v' following the walk $u'Q'PQv'$ in its small neighborhood in the interior f ; we cut C at its (two) intersection points with $\mathbf{p_u p_{v'}}$ and reconnected the severed ends on both sides by a curve following $\mathbf{p_u p_{v'}}$ in its small neighborhood thereby obtaining a closed curve, and a curve C' joining v' and u' . Finally, C' can be subdivided by vertices thereby yielding a desired embedding of $G \cup P'$. Otherwise, if ω_v is concave or $\mathbf{p_u p_{v'}}$ is crossed by an edge of G we need to deform the clustered embedding of G so that this is not longer the case.

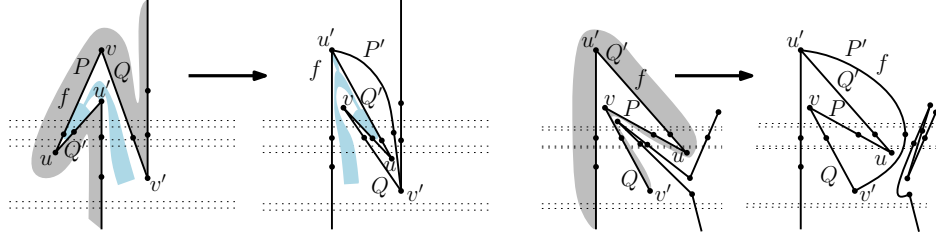


Fig. 5: A pair of deformations of the clustered embedding of G so that f can be subdivided by P' . For the sake of clarity clusters are drawn as horizontal strips rather than wedges.

By a *spur* with the *tip* u we understand a closed curve obtained as a concatenation of a line segment contained in a ray separating clusters and a curve contained in the boundary of f passing through exactly one extreme u of f such that the curve is longest possible. The *length* of the spur is one plus the number of its crossings with rays separating clusters divided by two. If ω_u is concave, the vertex u is a tip of a spur whose length is the distance of u to a closest other extreme along the face. Note that both P and Q' must be paths in this case. The rough idea in the omitted part of the proof is that shortest spurs have room around them to be deformed while maintaining the embedding clustered such that P' can be added. Spurs are deformed as illustrated in Fig. 5. (see Appendix for the rest of the proof) ■

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Appendix

Fan drawings

We show that the clusters can be drawn as regions, each bounded by a pair of rays emanating from the origin. Suppose that $G = (V_0 \uplus \dots \uplus V_{c-1}, E)$ is given by a clustered embedding living in the xy plane of \mathbb{R}^3 . We assume that boundaries of discs representing clusters do not touch. Consider a stereographic projection from the north pole of a two-dimensional sphere S sitting at the origin of \mathbb{R}^3 . Let D be a stereographical pre-image of the embedding of G on S . Let S' denote the union of G (as a topological space) with the boundaries of the clusters in D . Let R_n and R_s be a connected component of the complement of S' in S , respectively, containing the north pole and south pole. If necessary, we apply an isotopy to D (a continuous deformation keeping D to be a clustered embedding all the time) so that in the resulting embedding D of G on S every boundary of a cluster intersects (in fact touches) the closure of R_n and the closure of R_s .

We show that a desired isotopy exists. We contract every cluster to a point thereby treating clusters as vertices in an embedding D' of a cycle C of length c having multi-edges. Formally, this can be viewed as a quotient S/\sim , where $x \sim y$ iff x and y belong to the same cluster. In D' there must be a pair of faces f and f' whose facial walk is C since any cycle in the corresponding multi-graph is obtained as a symmetric difference of facial walks. Apply an isotopy to D' such that f contains the north pole in its interior and f' contains the south pole in its interior. Finally, we decontract clusters in the end. The above procedure can be easily turned into an isotopy of D .

By projecting the resulting spherical embedding back to the plan we can also assume that we have a clustered embedding of G such that clusters are represented by small discs of diameter $\epsilon > 0$ each drawn in a close vicinity of a different vertex of a regular convex c -gon with the center at the origin, and the edges between clusters V_i and $V_{i+1 \bmod c}$, for every i , are closely following the edge of the c -gon between the corresponding pair of vertices. The desired rays bounding clusters are those from the origin orthogonal to the sides of the c -gon.

Normalization

Proof (the omitted part of the proof from Section 3.2).

First, we suppose that ω_u is concave. W.l.o.g. we assume that $\mathbf{p}'_v \notin \mathbf{p}_v \mathbf{p}_u$. This holds when ω_v is convex, Figure 8a. Otherwise, we exchange the roles of u and v , see Figure 6. Combinatorially, there are two cases depending on whether v is concave, but we treat them simultaneously. We isolate a part of the embedding of G inside a spur represented by a topological disc D . In order to get a desired deformed clustered embedding of G we define a homeomorphism from D that we use to redraw the corresponding part of G thereby disconnecting some edges that are reconnected in the end. Let D_0 denote the topological disc bounded by the closed curve obtained by concatenating the line segment $\mathbf{p}_v \mathbf{p}_u$ with the parts of

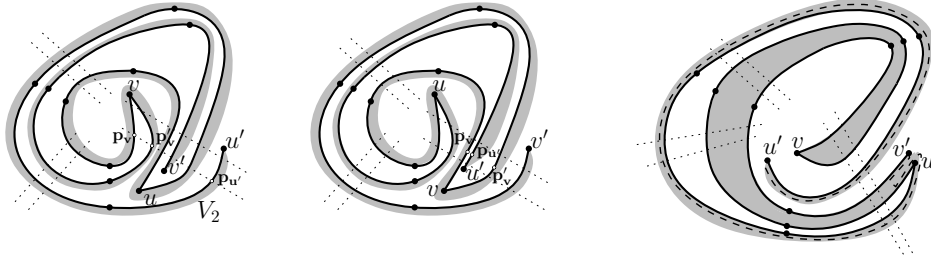


Fig. 6: Deformation in the case when both u and v have concave wedges incident to f that is indicated by grey. The dashed curve represents the path P' subdividing f . On the left, point $\mathbf{p}'_v \in \mathbf{p}_v \mathbf{p}_{u'}$. In the middle, point $\mathbf{p}'_v \notin \mathbf{p}_v \mathbf{p}_{u'}$. On the right, the corresponding deformation.

P and Q' connecting endpoints of $\mathbf{p}_v \mathbf{p}_{u'}$ with u . We assume that $v' \notin D_0$ which holds automatically when ω_v is concave due to $\mathbf{p}'_v \notin \mathbf{p}_v \mathbf{p}_{u'}$.

If v does not have a concave wedge incident to f it could happen that the boundary of D_0 crosses itself. As we will see later due to this reason we cannot simply put $D := D_0$. We use a “covering map” g , see Figure 7a. In the light of Jordan-Schönflies theorem, let a topological disc D be a pre-image of a continuous map $g : D \rightarrow D_0$ such that the map g is injective when restricted (in the target) to the embedding of G ; g maps the boundary of D to the concatenation of $\mathbf{p}_v \mathbf{p}_{u'}$ with the parts of P and Q' connecting endpoints of $\mathbf{p}_v \mathbf{p}_{u'}$ with u ; and the pre-image (of parts) of the rays separating clusters consists of a union of a connected part of the boundary of D (contained in the pre-image of $\mathbf{p}_v \mathbf{p}_{u'}$) and a set of pairwise disjoint diagonals of D . Treating G as a topological space let $G|_D := g^{-1}(G)$.

If G contains cut-vertices D can have pairs of boundary points identified, see Figure 7b. Let ℓ'_0 denote a line segment contained inside the intersection of the interior of f with a ray separating clusters containing \mathbf{p}'_v , whose end vertex is very close to \mathbf{p}'_v , and $\ell'_0 \subset \mathbf{p}'_v \mathbf{p}_v$ if and only if ω_v is convex. Let D' denote a disc bounded by ℓ'_0 and a curve joining the endpoints of ℓ'_0 following Q towards v' and back in its small neighborhood in the interior of f . Let ℓ_0 be the connected component on the boundary of D in $g^{-1}(\mathbf{p}_v \mathbf{p}_{u'})$. Let ℓ_i for $0 < i < |\text{height}(P)|$, denote a connected component of a pre-image by g of (a part of) a ray separating clusters. The segments ℓ_i 's are indexed by the order of appearance of their endpoints along the boundary of D . A line segment ℓ_i is just a point if it joins identified boundary points in D . Similarly, let ℓ'_i for $0 < i < |\text{height}(P)|$, denote a connected component of the intersection of D' with rays separating clusters. We contract ℓ'_i to a point iff ℓ_i is a single point and we contract interior parts of D' in the correspondence with D and ℓ_i 's as illustrated in Figure 7b. Since $v' \notin D_0$, we have $(g(D) = D_0) \cap D' = \emptyset$.

We map by a homeomorphism h the disc D to D' so that ℓ_i is mapped to ℓ'_i for all $0 \leq i < |\text{height}(P)|$ and so that the endpoint \mathbf{x} of ℓ_0 for which $g(\mathbf{x})$ is

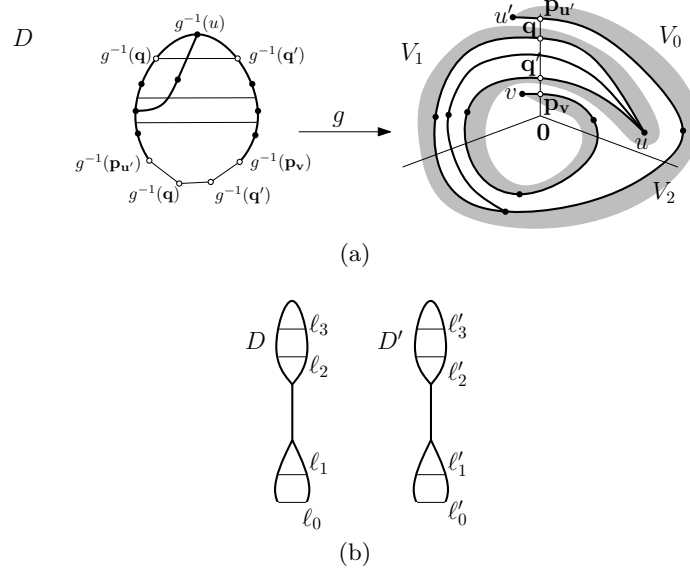


Fig. 7: (a) A “covering map” g from a disc onto a spur. (b) Identifying parts of the boundary of the discs D' according to D .

closer to ℓ'_0 is mapped to the point of ℓ'_0 closest to (farthest from) $g(\mathbf{x})$ if ω_v is concave (convex), see Figure 5 for an illustration. We alter the embedding of G by deleting $g(G|_D)$ and replacing it by $h(G|_D)$. Finally, we reconnect the severed end pieces of edges intersecting $\mathbf{p}_v\mathbf{p}_u$ by curves inside the cluster containing v without creating any edge crossing.

If $v' \in D_0$, we redraw the portion of G contained in the disc D (we override the previous D) bounded by the line segment $\mathbf{p}_v\mathbf{p}_u$ and the parts of P and Q joining end points of $\mathbf{p}_v\mathbf{p}_u$ with v , see Figure 10a. Note that the boundary of the disc D is non-self intersecting. If Q' does not cross $\mathbf{p}_v\mathbf{p}_u$ we make $\mathbf{p}_v\mathbf{p}_u$ crossing free by mapping the part of G in D to a long skinny disc D' (we override the previous D') in the vicinity of Q' and reconnecting the severed edges similarly as above.

If Q' crosses $\mathbf{p}_v\mathbf{p}_u$ we cannot apply the previous argument since $D \cap D' \neq \emptyset$, and thus, we need a different approach. We temporarily add to G a path $u'Q''w$ starting at u' , following closely Q' in the interior of f and ending in a vertex w . Let \mathbf{p}_w denote the crossing point of the last edge on Q'' separating clusters with a ray separating clusters. We cut edges at $\mathbf{p}_v\mathbf{p}_u$ by removing a small $\epsilon > 0$ neighborhood of their crossing points. We reroute the paths Q' (see Figure 9) and Q'' without crossing an edge of G from their severed ends outside of D past u and along Q' in the interior of f so that we closely follow the boundary of f . We easily avoid creating crossings since we cut edges at $\mathbf{p}_v\mathbf{p}_u$. Let D'' be the disc bounded by rerouted parts of Q' and Q'' from their first intersections on the way from u and w , respectively, with $\mathbf{p}_w\mathbf{p}'_u$ to their common vertex u' ; and by $\ell''_0 \subset \mathbf{p}_w\mathbf{p}'_u$

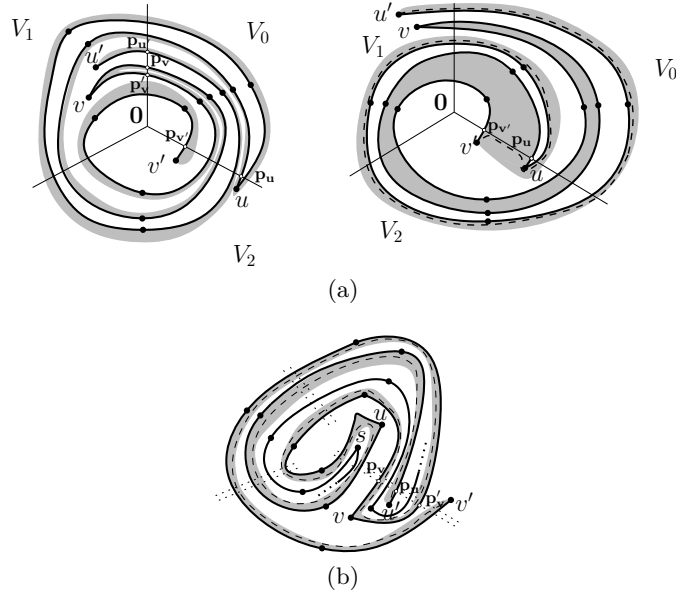


Fig. 8: (a) Deformation in the case when only u has a concave wedge incident to f that is indicated by grey. The dashed curve represents the desired path subdividing f . On the left, the line segment $\mathbf{p}_v\mathbf{p}_u$ crosses edges of G . On the right, the corresponding deformation. (b) The wedge at u in f , that is indicated by grey, is convex. A sink s in the interior of a spur having the vertex u as the tip.

connecting the ends of those parts. Let ℓ''_i , $0 \leq i < |\text{height}(P)| - c$, denote the line segments in the intersection of D'' with rays separating clusters listed in the order of appearance along the boundary of D'' . We deform the embedding of G by a mapping g from $D \setminus (P \cup Q)$, see Figure 10b for an illustration, such that g maps the relative interior of $\ell_0 = \mathbf{p}_v\mathbf{p}_u$ to the relative interior of $\mathbf{p}_w\mathbf{p}'_u$, the parts of Q' and Q'' in D to their rerouted counterparts, and for the open line segments ℓ_i , $1 \leq i < |\text{height}(P)| - c$, in the intersection of D with the rays separating clusters we have $g(\ell_{i+c}) \subset \ell''_i \subset g(\ell_i)$. The mapping g is then extended to the whole $D \setminus (P \cup Q)$ such that (i) g is a homeomorphism when restricted to the interior of the slab between ℓ_i and ℓ_{i+1} , for all $0 \leq i < |\text{height}(P)|$, where $\ell_{|\text{height}(P)|}$ is the final piece of the boundary of D'' , and (ii) no edge crossings are introduced, i.e., $g(D|_G)$ is injective and $g(D|_G) \cap G = \emptyset$. It is not hard to see that g exists. Finally, we reconnect the severed end pieces of edges inside the cluster containing u and remove Q'' . The previous deformation is perhaps easier seen as follows. The clustered embedding of $G|_D$ can be made arbitrarily skinny. Thus, for the purpose of deformation, we can picture that $G|_D$ consists just of the part of Q' and Q'' in D and a strictly monotone path starting at u' as in Figure 10a. We map $G|_D$ as indicated in Figure 10b.

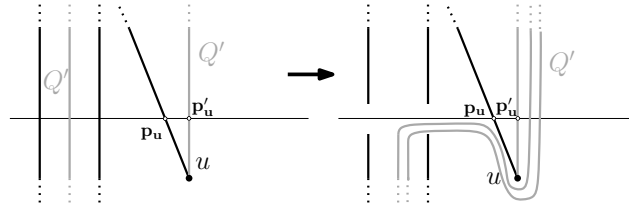


Fig. 9: Re-routing Q' past u along the boundary of f .

Second, if both ω_u and ω_v are convex we can subdivide f by P' unless $\mathbf{p}_v\mathbf{p}_{u'}$ is intersected by edge(s) of G (we still assume that $\mathbf{p}'_v \notin \mathbf{p}_v\mathbf{p}_{u'}$). However, if this is the case D_0 (defined above) contains a sink or source s in its interior, see Figure 8b. Note that we can assume that s is also a tip of a shortest spur of f , and hence, the previous case applies. ■

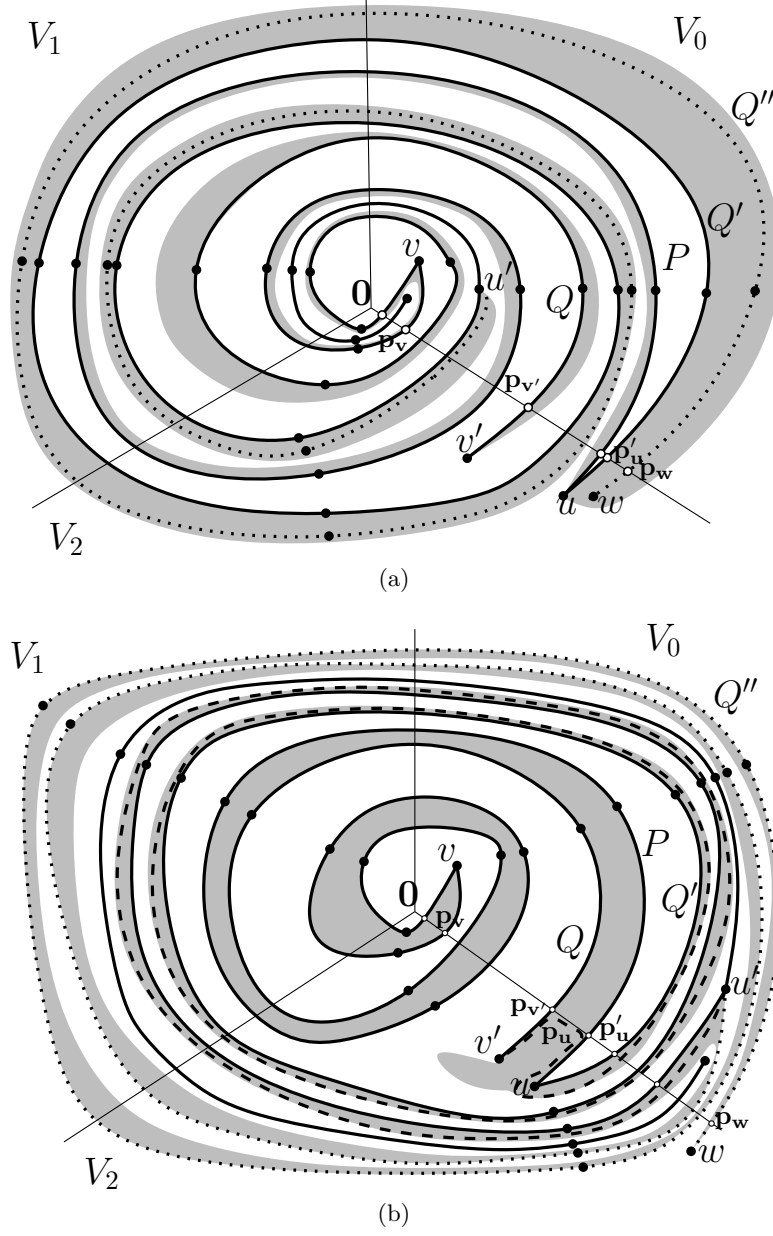


Fig. 10: The face f is indicated by grey. (a) The vertex $v' \in D_0$ corresponding to the spur with the tip u . (b) Corresponding deformation involving Q' and the dashed path P' subdividing f .